System Identification 1

arma models

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SYSTEM IDENFIFICATION 1: ARMA MODELS

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SYSTEM IDENTIFICATION 1: ARMA MODELS

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ABSTRACT

System identification by ARMA models has been investigated in theory and in practice. The theoretical relations between an ARMA model and the eigen frequencies and damping ratios of a linear vibrating system have been recognized. The method has been applied in practice to experimental data for a heavily damped SDOF system and a lightly damped two-degrees-of- freedom system. Different ARMA models have been estimated and the proper model order has been discussed. Two procedures with and without bandpass filtering of a single eigen mode have been applied. The results have shown that bandpass filtering is advantageous because it significantly reduces the proper model order. Mean values and standard deviations of eigen frequencies and damping ratios have been computed by simulation of the sample space of the AR-parameters. This was a practicable task when the models were models of low order such as ARMA(2,1) and ARMA(4,3) models.

1 INTRODUCTION: ARMA MODELS IN GENERAL

There is a need for system identification methods which provide reliable estimates of eigen frequencies and damping ratios of vibrating structures in civil engineering. The typical identification problem is that the measurement of a structure only provides reliable information about the structural response while the forces due to wind or waves are only indirectly known due to some theoretical model. In such cases the general assumption is that the input forces excite each structural resonance as if the input forces were white noise. This means that the response spectra at resonance peaks represents the shape of the transfer function of the structure. Hence the eigen frequencies and the damping ratios can be estimated due to this so called local or equivalent white noise assumption.

The identification problem is to obtain a reliable model (or transformation) of the response, e.g. the response spectrum and extract information about the modal parameters. In such cases the application of ARMA models exists as an alternative to the conventional combination of FFT analysis and curve fitting algorithms.

The point of the aplication of ARMA models is that the FFT analysis only provides a non-parametric spectrum while the ARMA model gives a parametric model given by a model in the frequency domain or a model in the time domain. This gives the ARMA model the following advantages :

1. The eigen frequencies and the damping ratios can be estimated directly from the parameters in the ARMA model if the proper relations are known.
2. If the uncertainties of the ARMA parameters are given by a covariance matrix then the confidence intervals can be computed for the eigen frequencies and damping ratios.
3. The numerical inaccuracies of the FFT analysis due to insuffient frequency resolution and leakage (windowing) are avoided by the application of ARMA models. However to avoid aliasing the sampling frequency of course still has to be greater than the Nyquist frequency.

The disadvantage of the ARMA model is that it requires some computer capacity because the estimation of the ARMA parameters is a non-linear problem.

The basic properties of an ARMA model is presented in the present chapter, further explanations are e.g. given in Box- Jenkins [1] and Pandit and Wu [2]. The estimation procedure of ARMA models is briefly described in chapther 2. In chapter 3 the ARMA model for a vibrating system is derived. And finally in chapter 4 identification by ARMA models is illustrated by two experimental examples. Chapter 4 can be read without having read chapter 3.1 which contains some mathematics.

1.1 The ARMA Model

An ARMA model is a parametric statistical model of a time series of some random data. Any stationary time series can be considered as the output of some time invariant black box model with stationary normally distributed white noise as input, see figure 1.1. Hence the black box model transforms the white noise into some specific random noise time series.

The white noise, $a(t)$ is defined by:

$$E[a(t)] = 0 \quad (1.1)$$
$$E[a(t)a(t+s)] = 2\pi S_0 \delta(s) \quad (1.2)$$

where S_0 is a real constant and $\delta(s)$ is diracts delta function.

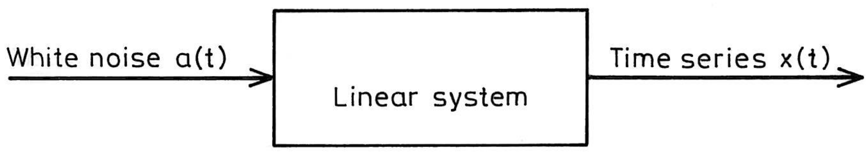

Figure 1.1. Black box model.

The black box is assumed to be a linear time invariant system which means that the measured time series is assumed to be normally distributed. Throughout the text $x(t)$ is assumed to have a zero mean:

$$E[x(t)] = 0 \quad (1.3)$$

The time series is assumed to be obtained with constant sampling of a continuous signal:

$$x_t \equiv x(t\Delta), \quad t = 1, 2, 3 \ldots \quad (1.4)$$

The ARMA model of the time series now given by:

$$x_t = \underbrace{\sum_{i=1}^{n} \Phi_i x_{t-i}}_{AR-part} + a_t - \underbrace{\sum_{i=1}^{m} \Theta_i a_{t-i}}_{MA-part} \quad (1.5)$$

This is called an ARMA(n,m) model. The parameters in the ARMA model are real numbers. It is seen that x_t is expressed as a linear combination of the x_ts and a_ts of the past plus of course a contribution from the white noise at the time t. The sum of the x_ts of the past refers to the autoregressive part of the ARMA model (the AR part) and the sum of the a_ts of the past refers to the moving average part of the ARMA model (the MA part). Hence the AR parameters are the Φs and the MA parameters are the Θs.

The discrete white noise a_t is defined by:

$$E[a_t] = 0 \tag{1.6}$$
$$E[a_t a_{t+k}] = {\sigma_a}^2 \delta_k, \quad k > 0 \tag{1.7}$$

where δ_k is the Kronecker delta.

1.2 Green's Function

In (1.5) x_t was expressed as a linear combination of the x_ts and a_ts of the past, however x_t can also be solely expressed as a linear combination of the a_ts of the past. This is done with the Green's function, which is analoguous to the impulse response function defined in the continuous time domain. The Green's function is defined by:

$$x_t = \sum_{j=0}^{t} G_{t-j} a_j \tag{1.8}$$
$$G_0 = 1 \tag{1.9a}$$
$$G_j = g_1 \lambda_1^j + g_2 \lambda_2^j + \ldots + g_n \lambda_n^j \tag{1.9b}$$

The λ_j can be shown to be the root of:

$$\lambda^n - \Phi_1 \lambda^{n-1} - \Phi_2 \lambda^{n-2} - \ldots - \Phi_n = 0 \tag{1.10}$$

and g_i is defined by:

$$g_i = \frac{\lambda_i^m - \Theta_1 \lambda_i^{m-1} \cdots - \Theta_m}{(\lambda_i - \lambda_1)(\lambda_i - \lambda_2)\cdots(\lambda_i - \lambda_{i-1})(\lambda_i - \lambda_{i+1})\cdots(\lambda_i - \lambda_n)} \tag{1.11}$$
$$i = 1, 2, 3 \ldots, n$$

(1.10) is a very important equation because if the Φs are known the λs can be estimated and from the λ_is a relation can be establish to a differential equation in the continuous time domain. This means that the dynamic parameters of a vibration system such as eigen frequencies and damping ratios can be determined directly from the estimated Φs. This will be shown in chapter 3 for the multivariate case.

1.3 The Inverse Function

Another important function is the inverse function which is analoguous to the Greens function but with the important difference that the black box model in figure 1.1 has now been changed to having an input signal x_t and an output signal a_t. The inverse function I_j is indirectly defined by:

$$a_t = \sum_{j=0}^{t} -I_j x_{t-j} \tag{1.12}$$

$$I_0 = -1 \tag{1.13a}$$

$$I_j = -i_1\nu_1^j - i_2\nu_2^j \cdots - i_m\nu_m^j \tag{1.13b}$$

where ν_i is determined from:

$$\nu^m - \Theta_1\nu^{m-1} - \Theta_2^{m-2} \cdots - \Theta_m = 0 \tag{1.14}$$

and i_i is found from:

$$i_i = \frac{\nu_i^n - \Phi_1\nu_i^{n-1} - \cdots - \Phi_n}{(\nu_i - \nu_1)(\nu_i - \nu_2)\cdots(\nu_i - \nu_{i-1})(\nu_i - \nu_{i+1})\cdots(\nu_i - \nu_m)} \tag{1.15}$$
$$i = 1, 2, 3 \ldots, m$$

The equations are seen to be very similarly to the equations of the Green's function due to the analogy.

1.4 Stability and Invertibility

With respect to the ARMA parameters of a real structure these are the same conditions analoguous to the stability conditions of a system of differential eqautions for a real structure. These are called stability and invertibility conditions.

From (1.8) and (1.12) it is seen that the following conditions have to be satisfied:

$$|\lambda_k| < 1 \tag{1.16}$$

$$|\nu_k| < 1 \tag{1.17}$$

(1.16) and (1.17) mean that a finite input will always be uniquely related to a finite output and vice versa, i.e. x_t or a_t will not increase without bounds. (1.16) is called the stability conditions and (1.17) is called the invertibility conditions. (1.16) and (1.17) can be replaced by implicit expression of Φs and Θs due to (1.10) and (1.14). The conditions are important because they have to be satisfied when an ARMA model is estimated. It should be mentioned that sometimes the stability conditions are called stationary conditions (Box and Jenkins [1], NAG fortran library [3]).

1.5 The Auto Covariance

The autocovariance function of x_t is perhaps the most important property of an ARMA model. It is often demanded that the covariance function of an ARMA model has the same form as the assumed time continuous theoretical model of the time series. The appropriate form of the covariance function is obtained by a proper choice of the order of the ARMA model given by the AR order, n and the MA order, m. The covariance equivalence condition ensures that the statistical moments up to the second order of the time series are reproduced in the ARMA model in the most accurate way.

The auto covariance function of the ARMA(n,m) model is given by:

$$\gamma_k = \sum_{i=1}^{n} d_i \lambda_i^k \tag{1.18}$$

where d_i is given by:

$$d_i = \sigma_a^2 \sum_{j=1}^{n} \frac{g_i g_j}{1 - \lambda_i \lambda_j} \qquad i = 1, 2, 3 \ldots, n \tag{1.19}$$

g_i is given by (1.10) and (1.11). As it will be shown in chapter 2 the ARMA models are estimated by minimizing the variance of the assumed input, σ_a^2. This means that the variance of the auto covariance function given by (1.19) is minimized. Consequently the estimated auto covariance function should be a more reliable estimate than the sampled auto covariance function given by:

$$\hat{\gamma}_k = \frac{1}{N} \sum_{t=k+1}^{N} x_t x_{t-k} \tag{1.20}$$

Thus, the importance of ARMA models is seen from the fact that the ARMA model provides a more reliable time series analysis than the FFT analysis which is based on the sampled auto covariance function.

2 ESTIMATION STRATEGY

The model order of the ARMA model is in general chosen as (*2n*,*2n-1*) because an ARMA(*2n*,*2n-1*) model is covariance equivalence with a vibrating linear system with n degrees of freedom. This is shown in chapter 3. It should also be noted that an even AR order means that the roots of (1.10) will be complex in general, corresponding to a underdamped vibrating system as shown in chapter 3.2. These model considerations mean that generally the AR order should be chosen to be even.

Finally it can be shown, see Pandit and Wu [2], that any time series can be fitted to an ARMA(*2n*,*2n-1*) model provided that the order *2n* is chosen sufficiently high. This certain type of model leads to a too high AR or MA order which can lead to a distortion of the deterministic system characteristics hidden in the signal. This is known as overfitting and means that the random noise characteristics are incorporated in the model of the system which leads to a distorted model. The proper choice of the model order is consequently an important pratical problem in the application of ARMA models in the field of system identification.

2.1 Parameter Estimation

The parameters of the ARMA model have to be estimated from the time series x_t only. This is done by minimising the criterion function which in the present paper is identical with the computed variance of the white noise input, a_t:

$$\sigma_a^2(\Phi_i, \Theta_j, X_t) = \frac{1}{N}\sum_{t=1}^{N} a_t^2 \tag{2.1}$$

This is also sometimes called the residual sum of squares (RSS) or the loss function. The minimising can be done by a recursive or batch algorithm. According to Lefkowitz [5], the most efficient methods of each type are the RPEM method (Recursive Prediction Error Method), see Ljung [4], and the Box and Jenkins Maximum Likelihood algorithm, see Box and Jenkins [1]. In this paper both algorithms have been used in the derivation of the ARMA models.

The subject of the least square problem is to minimise a function which is nonlinear with respect to the ARMA parameters. The nonlinear nature of the problem is seen from the rewriting of (1.6) where n is replaced by $2n$:

$$a_t = x_t - \sum_{i=1}^{2n} \Phi_i x_{t-i} - \sum_{i=1}^{2n-1} \Theta_i a_{t-i} \tag{2.2}$$

where a_t becomes a non-linear function of the ARMA parameters if a_{t-i} is eliminated from the expression by the equation itself. The non-linear least square problem is usually solved by Marquardts algorithm, which is related to the Steepest Decent method and the Newton method, see Marquardt [6].

(2.2) is limited to the case where $t \geq 2n+1$ because x_{-1}, x_{-2} etc. is unknown. Therefore the initial values:

$$a_j = 0, \quad j = 0, 1, 2, 3 \ldots, 2n \tag{2.3}$$

are applied. Some initial estimates of the ARMA parameters are also necessary because a_t has to be computed in (2.1) and (2.2). Different methods are available, see Box and Jenkins [1] and Pandit and Wu [2], but it might be sufficient to chose estimates equal to zero.

3 THE SYSTEM OF n DEGREES OF FREEDOM

In Pandit and Wu [2] it is shown that an ARMA(2,1) model is consistent with a linearly vibrating one-degree-of-freedom system. Meaning that there is equivalence with respect to the covariance between the system and the model. From this special case it follows intuitively that a system with n degrees of freedom corresponds to an ARMA(*2n,2n-1*) model. This is logical because a system of n degrees of freedom consists of n eigen modes and each mode can be considered as a single-degree-of-freedom system. Hence n degrees of freedom correspond to the ARMA(*2n,2n-1*) model. This can also be shown and the mathematical relation can be established between the parameters in the differential equations of the vibrating system and the ARMA model. This will be the subject in this chapter, for further reference, see Shinozuka et al. [7] and Kozin and Natke [8].

3.1 Derivation of The ARMA Model

In the continuous time domain a vibrating system is described by a system of linear differential equations of second order with constant coefficient given by a mass matrix $\overline{\overline{M}}$, a stifness matrix $\overline{\overline{K}}$, a damping matrix $\overline{\overline{C}}$, a force distribution vector $\overline{f}$ and a white noise input $a(t)$:

$$\overline{\overline{M}}\ddot{\overline{z}} + \overline{\overline{C}}\dot{\overline{z}} + \overline{\overline{K}}\overline{z} = \overline{f}a(t) \tag{3.1}$$

(3.1) can be rewritten in the state space form by the transformation $\overline{x}^T = (\overline{z}^T, \dot{\overline{z}}^T)$:

$$\dot{\overline{x}}(t) = \overline{\overline{A}}\,\overline{x}(t) + \overline{b}\,a(t) \tag{3.2}$$

where $\overline{\overline{A}}$ is a $(2n \times 2n)$ matrix given by:

$$\overline{\overline{A}} = \begin{pmatrix} \overline{\overline{0}} & \overline{\overline{I}} \\ -\overline{\overline{M}}^{-1}\overline{\overline{K}} & -\overline{\overline{M}}^{-1}\overline{\overline{C}} \end{pmatrix}$$

$$\overline{b} = \begin{pmatrix} \overline{0} \\ \overline{\overline{M}}^{-1}\overline{f} \end{pmatrix}$$

The measured response is given by:

$$\overline{y} = \overline{\overline{H}}\,\overline{x} \tag{3.3}$$

where $\overline{\overline{H}}$ is the transformation matrix between the response of the measured degrees of freedom and the number of degrees of freedom assumed to be represented by eigen modes in the measured response $y(t)$.

The solution of (3.2) can be shown to be:

$$\overline{x}(t) = \exp(\overline{\overline{A}}(t-s)) + \int_s^t \exp(\overline{\overline{A}}(t-\tau))\overline{b}\,a(\tau)\,d\tau \tag{3.4}$$

where $\exp(\overline{\overline{A}}t)$ is defined by the expansion:

$$\exp(\overline{\overline{A}}t) \equiv \overline{\overline{I}} + (\overline{\overline{A}}t) + \frac{(\overline{\overline{A}}t)^2}{2!} + \frac{(\overline{\overline{A}}t)^3}{3!} + \cdots \tag{3.5}$$

Now if the time is discretised, $t = (k+1)\Delta$, $s = s\Delta$, $x(k\Delta) \equiv x_k$ the following solution is obtained :

$$\overline{x}_{k+1} = \exp(\overline{\overline{A}}\Delta)\,\overline{x}_k + \int_{k\Delta}^{(k+1)\Delta} \exp(\overline{\overline{A}}((k+1)\Delta - \tau))\,\overline{b}\,a(\tau)\,d\tau \tag{3.6}$$

The $(2n \times 2n)$ matrix, $\exp(\overline{\overline{A}}\Delta)$ has the characteristic equation:

$$\det(\exp(\overline{\overline{A}}\Delta) - \lambda\overline{\overline{I}}) = 0 \tag{3.7}$$

which is given by the equation:

$$p^{2n} + \beta_{2n-1}p^{2n-1} + \cdots + \beta_1 p + \beta_0 = \phi(p) = 0 \tag{3.8}$$

(3.8) is used in the derivation of the expression of the ARMA model for an n degrees of freedom system. (3.8) is seen to be analogous to (1.10).

The measured response vector, $\overline{y}$ obtained by (3.2) can now be expressed by the solution (3.5) for $2n$ sampling points corresponding to the number of equations in (3.2):

$$\begin{aligned}
\overline{y}_k &= \overline{\overline{H}}\overline{x}_k \\
\overline{y}_{k+1} &= \overline{\overline{H}}\exp(\overline{\overline{A}}\Delta)\overline{x}_k + \overline{\overline{H}}\overline{u}_k \\
\overline{y}_{k+2} &= \overline{\overline{H}}\exp(\overline{\overline{A}}2\Delta)\overline{x}_k + \overline{\overline{H}}\overline{u}_k\exp(\overline{\overline{A}}\Delta) + \overline{u}_{k+1} \\
\cdots &= \cdots \\
\cdots &= \cdots \\
\cdots &= \cdots \\
\overline{y}_{k+2n} &= \overline{\overline{H}}\exp(\overline{\overline{A}}2n\Delta)\overline{x}_k + \sum_{i=0}^{2n-1}\overline{\overline{H}}\overline{u}_{k+i}\exp(\overline{\overline{A}}(2n-1-i)\Delta)
\end{aligned} \tag{3.9}$$

where $\overline{u}_{k+l}$ is given by:

$$\overline{u}_{k+l} = \int_{(k+l)\Delta}^{(k+l+1)\Delta} \exp(\overline{\overline{A}}((k+l+1)\Delta - \tau))\overline{b}a(\tau)d\tau \tag{3.10}$$

Upon multiplying each $\overline{y}_{k+j}$ by β_j, $j = 0, 1\ldots, (2n-1)$ and having added all the equations together the following is obtained :

$$\overline{y}_{k+2n} + \beta_{2n-1}\overline{y}_{k+2n-1} + \cdots + \beta_1\overline{y}_{k+1} + \beta_0\overline{y}_k =$$

$$\overline{\overline{H}}(\beta_0 + \beta_1 \exp(\overline{\overline{A}}\Delta) + \beta_2 \exp(\overline{\overline{A}}2\Delta) + \cdots + \exp(\overline{\overline{A}}2n\Delta))\overline{x}_k$$

$$+\beta_1\overline{\overline{H}}\overline{u}_k + \beta_2\overline{\overline{H}}(\overline{u}_k \exp(\overline{\overline{A}}\Delta) + \overline{u}_{k+1}) + \cdots\cdots$$

$$+\beta_{2n-1}\overline{\overline{H}}(\overline{u}_k \exp(\overline{\overline{A}}(2n-2)\Delta) + \overline{u}_{k+1} \exp(\overline{\overline{A}}(2n-3)\Delta) + \cdots + \overline{u}_{k+2n-2})$$

$$+\overline{\overline{H}}(\overline{u}_k \exp(\overline{\overline{A}}(2n-1)\Delta) + \overline{u}_{k+1} \exp(\overline{\overline{A}}(2n-2)\Delta) + \cdots + \overline{u}_{k+2n-1}) \qquad (3.11)$$

According to the Cayley Hamilton theorem it follows that the characteristic polynomial, $\phi(p)$ satisfies the equation:

$$\phi(\exp(\overline{\overline{A}}\Delta)) = 0 \qquad (3.12)$$

which leads to the reduction of (3.11):

$$\overline{y}_{k+2n} + \beta_{2n-1}\overline{y}_{k+2n-1} + \cdots + \beta_1\overline{y}_{k+1} + \beta_0\overline{y}_k =$$
$$\alpha_{2n-1}\overline{u}_{k+2n-1} + \alpha_{2n-2}\overline{u}_{k+2n-2} + \cdots + \alpha_0\overline{u}_k \qquad (3.13)$$

Here the coefficients α_{2n-i} are here the constants found by rewriting (3.11) in terms of $\overline{u}_{k+2n-i}$. It is seen that (3.13) is a multi-dimensional ARMA(*2n,2n- 1*) model if and only if $\overline{u}_k$ given by (3.10) is white noise. From the fact that $a(t)$ is white noise with zero mean it follows from the definition (3.10) that $\overline{u}_k$ is white noise with zero mean and the variance:

$$E[\overline{u}_k\overline{u}_k^T] = \sigma_a^2 \int_0^\Delta \exp(\overline{\overline{A}}\tau)\overline{b}\overline{b}^T \exp(\overline{\overline{A}}^T\tau)d\tau \qquad (3.14)$$

Thus (3.13) is an ARMA model. It is shown in appendix A that this ARMA(*2n,2n-1*) model has exactly the same covariance properties as the response of the model given by the system of differential equations, (3.1). The reproduction of the correct statistical moments up to second order is a very important quality of this ARMA model because it provides a complete statistical description of the time series, since $x(t)$ has been assumed to be normally distributed.

The equivalence between the covariance function of the model in the continuous and discrete time domain can be found from the solutions (3.4) and (3.6), see Kozin and Natke [8].

3.2 Estimation of The Dynamic Parameters

After the ARMA model of a linear vibrating system with n degrees of freedom has been derived the last step is to determine the relationship between the parameters in the ARMA model and the dynamic parameters of the vibrating system. The dynamic parameters are here given by the eigen frequencies and the damping ratios. It follows from the derivation of the ARMA model that the determination of the dynamic parameters corresponds to the solution of the equations of motion for the damped system.

The dynamic parameters are found from the $2n$ eigen values of $\overline{\overline{A}}$ given by the diagonal matrix $(\, \mu_i \,)$ in the state space formulation:

$$\mu_{(i)12} = -\omega_i\zeta_i \pm \omega_i\sqrt{{\zeta_i}^2 - 1} \qquad \zeta_i \geq 1.0 \tag{3.15a}$$

$$\mu_{(i)12} = -\omega_i\zeta_i \pm \omega_i i\sqrt{1 - {\zeta_i}^2} \qquad \zeta_i < 1.0 \tag{3.15b}$$

Now the $2n$ eigen values of $\exp(\overline{\overline{A}}\Delta)$, given by the diagonal matrix $(\, \lambda_i \,)$ can be found from the characteristic polynomial :

$$p^{2n} + \beta_{2n-1}p^{2n-1} + \cdots + \beta_1 p + \beta_0 = \phi(p) = 0 \tag{3.8}$$

In appendix B it is shown that the eigen values found by (3.8) are identical with the values in the diagonal matrix $(\, \exp(\mu_i\Delta) \,)$:

$$(\, \lambda_i \,) = (\, \exp(\mu_i\Delta) \,) \tag{3.16}$$

This means that if the eigen values of (3.8) and the sampling interval Δ are known the eigen values of $\overline{\overline{A}}$ can be determined by:

$$\mu_{(i)12} = \ln(\lambda_{(i)12})/\Delta \tag{3.17}$$

Here the index (12) refers here to the fact that the λ_is are found as complex conjugated pairs if the modes are underdamped. Finally due to (3.15) the values of the corresponding eigen frequencies and damping ratios can be found. Hence the relationship between the AR parameters and the eigen frequencies and damping ratios has been established

Since the coefficients of the characteristic equation (3.8) are real it follows that the roots of (3.8) will occur as complex roots in conjugated pairs if the modes are underdamped. Otherwise the roots will be real values.

The n pairs of roots will correspond to the roots of n polynomials of the second order derived from the polynomial of order $2n$ given by (3.8). Consequently if only underdamped modes are considered each polynomial of second order will represent an eigen mode given by an eigen frequency and a damping ratio which can be found from the two roots of the polynomial.

3.3 Statistic Distribution

If the AR-parameters are assumed to be jointly normally distributed and if knowledge of the covariance matrix exists then confidence intervals of the dynamic parameters can be estimated. This is a very important feature of using ARMA models although some problems exist in evaluating the confidence intervals.

First of all it is not possible to establish an explicit density function of the dynamic parameters because they are determined implicitly from the AR-parameters due to the polynomial relationship (3.8). This means that the distribution of dynamic parameters can only be found by simulation of events in the sample space of the AR- parameters.

A simulation program has been developed and the required number of simulated events has been found from the convergence of the statistical estimates. Acceptable convergence for the estimates of mean values and standard deviations were found for about 10 sample points for each AR parameter in the model while convergence of the correlation coefficient between the eigen frequency and the damping ratio of a given mode required at least 100-200 points.

It is to be noticed that the eigen frequencies and the damping ratio cannot be expected to be normally or logarithmic normally distributed due to the non-linear relation to the AR parameters. Thus any approximation to a given distribution function has to be checked by e.g. a χ_i^2 test.

Another problem exists because of the normal distribution assumption. Remembering the stability conditions in chapter 1.3 the normal distribution does not work if the sample space of the parameters contains a subspace where the stability conditions are violated and the probability density is not insignificant. This problem will exist for time series which are close to becoming instable and are characterised by uncertain parameters. For a vibrating system this means e.g. lightly damped systems or systems with eigen frequencies close to zero. For such systems the sample space of the AR- parameters must be limited by the stability conditions, and a proper choice of a probability distribution of the dynamic parameters may be the logarithmic normal distribution.

4 EXPERIMENTAL EXAMPLES

Identification by ARMA models has been investigated with data obtained from experiments. Real data are important because they give an understanding of the individual methods and the degree of their interpretation in practice.

A heavily damped single-degree-of-freedom system and a lightly damped two-degrees-of-freedom system have been considered. These two cases provide a good insight in the basic nature of the identification method.

4.1 Heavily Damped SDOF System

From Stedudel [9] data of a mechanical system were provided. The data have been further analysed by Pandit and Wu [2] and by the author of this paper. The system was a mechanical system consisting of a mass (1.96 kg), a linear elastic spring (851.46 N/m) and a viscous damper (dashpot) (37.46 kg/s). The mass of the system was bombarded with random impulse shocks. These shocks were manually produced by drumming on the mass in a manner that was intended to produce an effect analogous to a random impulse function simular to white noise. The expected values of the eigen frequency and the damping ratio were:

$$f = 3.3184\,Hz$$

$$\zeta = 0.459$$

The response of the mass was measured by two strain gauges and sampled with time intervals of 0.02 seconds, 130 times corresponding to a time series of 2.6 sec. Different ARMA models have been estimated from the data and compared with the results from Pandit and Wu [2]. The applied estimation algorithm was based on Box-Jenkins ML-algorithm, NAG [3]. Pandit and Wu's estimations were based on a self designed least squares algorithm closely related to the method of Box-Jenkins.

The application of the NAG library was not without problems. Convergence of the ARMA parameters was often a problem. This resulted in missing standard deviations of the parameters and sometimes invalid estimates. The convergence problems aroused when models of high order were considered. The explanation may be that models of high order give rise to uncertain estimation because the extra degrees of freedom are due to noise in the time series.

The expected model of the time series obtained from the response of the system was an ARMA(2,1) model. But estimation of models of higher order showed that the model error, σ_a^2 given by (2.1) could be reduced further, see figure 4.1. The figure shows the results of Pandit and Wu [2] and the results of the author. The reduction in the variance is seen to decrease with the model order. The difference in the development of the two curves is probably due to different criteria of convergence. However, the estimated ARMA(2,1) and ARMA(4,3) models were almost identical with the estimated models in Pandit and Wu.

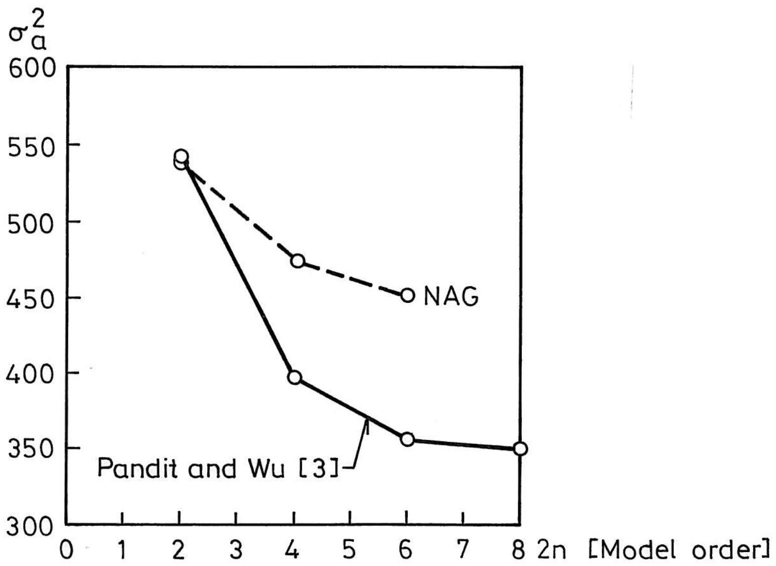

Figure 4.1. The reduction in the σ_a^2 with respect to the order of the model ARMA$(2n, 2n-1)$.

As a further investigation of the proper model order of the different models, they have been compared in the spectral form as shown in figure 4.2. This has been performed by transforming the ARMA models into the frequency domain. The transformation can be obtained by z-transformation which leads to the parametric expression of the autospectrum of an ARMA(n, m) model, see appendix C :

$$S_{xx}(f) = \Delta 2\sigma_a^2 \frac{\left|e^{mif2\pi\Delta} - \Theta_1 e^{(m-1)if2\pi\Delta} - \dots - \Theta_m\right|^2}{\left|e^{nif2\pi\Delta} - \Phi_1 e^{(n-1)if2\pi\Delta} - \dots \Phi_n\right|^2}, \quad 0 \leq f \leq \frac{1}{2\Delta} \tag{4.1}$$

Even though there is only small deviations in the estimated parameter values of the ARMA(4,3) models (about 5%) a disagreement is clearly seen in the spectral form. The two estimated ARMA(6,5) models deviate substantially from the models of lower order, especially the model estimated in Pandit and Wu [2]. Those disagreements can be explained by overfitting, see Ljung [4]. Overfitting means that the model error, σ_a^2 has been minimised by adapting the model further to the noise present in the time series. This gives rise to a violation of the parameter estimates. The overfitted model is also recognized by a very small reduction in σ_a^2, see figure 4.1. From figure 4.1 the correct order of the model can therefore be concluded to be an ARMA(4.3).

The eigen frequencies and the damping ratios have been computed directly from the AR-parameters of the models, see table 4.1. The estimated eigen frequencies and damping ratios are seen to agree within 6% and 11%, respectively. All the estimated values are within the 95% confidence interval of the ARMA(2,1) model.

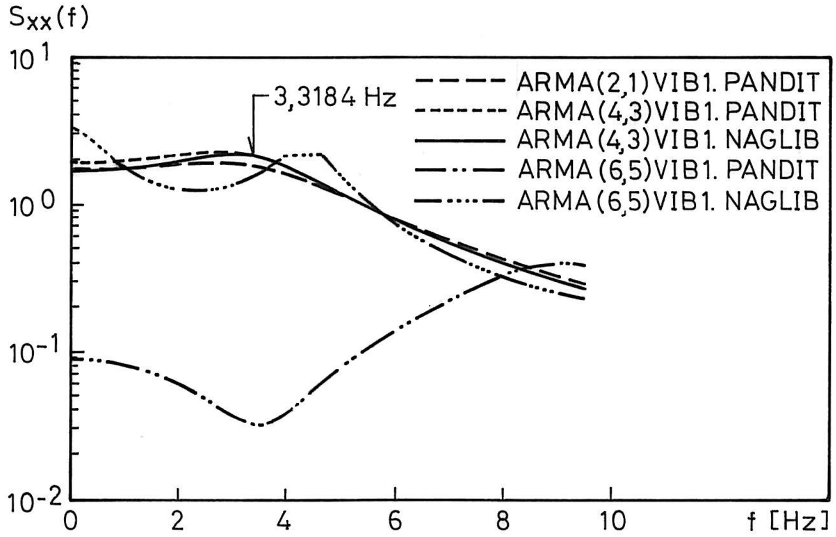

Figure 4.2 The autospectra of different ARMA models.

As shown in figure 4.2 the two estimation sessions of the ARMA(4,3) model give different estimates. This is probably due to different convergence criteria. The deviation of the results are about 1% for the eigen frequency and 13% for the damping ratio, see table 4.1. The deviation of the damping ratio is not satisfactorily. However, a proper choice could be the model estimated in Pandit and Wu [2] since it gives the smallest model error, see figure 4.1. Thus it is shown that it is necessary to be aware of the convergence criteria when ARMA models are being estimated.

The eigen frequency and the damping ratio of this model are seen to deviate 11% and 10% from the expected values, respectively. Those deviations may be due to the possibility that the theoretical parameters or model give an insufficient description of the real system.

The confidence intervals have been computed numerically for the ARMA(2,1) models assuming the eigen frequency and the damping ratio were normally distributed. The normal distribution has been applied since the variation coefficients of both parameters were found to be small. The simulated joint density function of the eigen frequency and the damping ratio is shown in figure 4.3. About 1600 points were more than sufficient to give reliable statistical estimates. However, in the figure 40000 points have been used to get a smooth graph. The statistical analysis showed that the eigen frequency and the damping ratio of the first eigen mode were uncorrelated.

	f_1	Hz	ζ_1	
Expected values	3.3184		0.459	
ARMA(2,1) computed from Pandit and Wu	3.8590	± 1.3113	0.5328	$\pm$ 0.2606
ARMA(4,3) computed from Pandit and Wu	3.6410	*)	0.4777	*)
ARMA(4,3) computed from estimates obtained by NAG	3.6871	*)	0.4140	*)

Table 4.1. Computed eigen frequencies and damping ratios from the estimated ARMA parameters with 95% confidence intervals. *) Confidence intervals have not been computed due to lack of statistical information about the ARMA parameters.

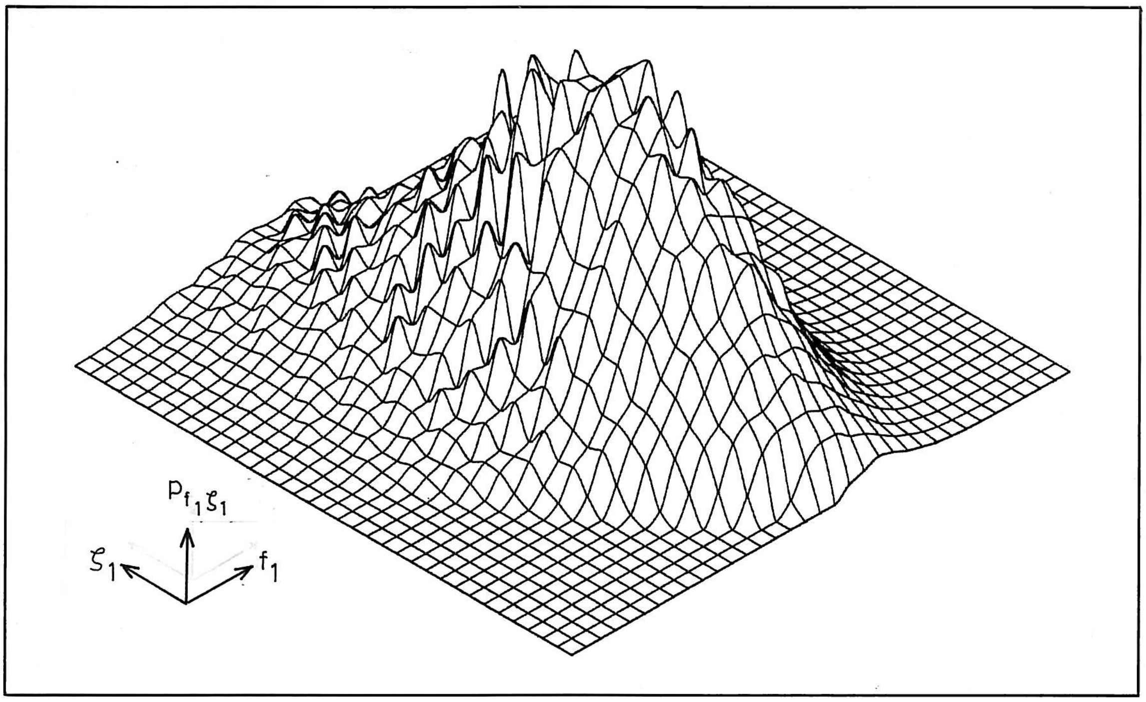

Figure 4.3 Simulated joint density function of the eigen frequency and the damping ratio.

4.2 Lightly Damped 2DOF System

A lightly damped system with two degrees of freedom was investigated, see figure 4.4. The system was a monopile with two concentrated masses. The two degrees of freedom assumption was justified by filtering the excitation and by appropriate unidirectional excitation of the monopile. The experiment is described in detail in Jensen [10].

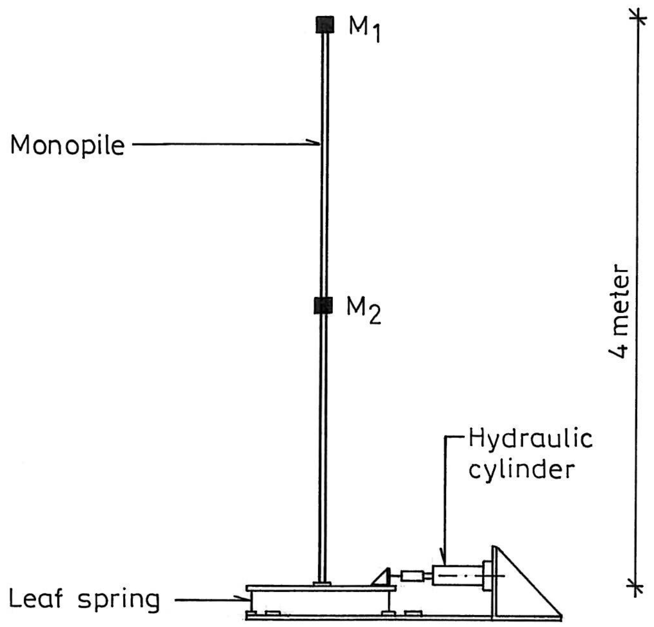

Figure 4.4. The monopile model and the excitation system.

The excitation was performed by a shaker table which was displacement controlled by a hydraulic system. The base displacement was white noise filtered by a lowpass filter with a cut-off frequency about 15 Hz and further filtered due to the effect of the hydraulic excitation system. The excitation force on mass no. i was given by:

$$p_{(i)}(t) = -M_{(i)}\ddot{u}_b(t) \tag{4.1}$$

where $\ddot{u}_b(t)$ was the acceleration of the base. This meant that a pure white noise displacement gave the force spectrum for the force on mass no. i:

$$S_{pp(i)}(f) = (2\pi f)^4 M_{(i)}^2 S_{u_b u_b}(f) \tag{4.2}$$

Clearly this was not a white noise spectrum. However, white noise was assumed locally around each resonance peak of the transfer functions because the slope of the input spectrum was assumed to be sufficiently flat compared with the sharpness of the resonance peaks.

It can be noticed that the force input and consequently the measured response was still normally distributed random signals as far as the system was linear and the input was normally distributed. This meant that the covariance equivalence property of the ARMA model still gave a sufficient statistical characterisation of the response.

Other contributions to the distortion of the time series applied to the ARMA modelling were the failure of the accelerometers at frequencies less than 0.9 Hz and the double integration of the accelerations to displacement signals. These contributions distorted the measured output and consequently also the estimated ARMA models. However the most important source to model error and uncertainty was without doubt the non-white force excitation.

Two procedures of ARMA modelling were performed. The purpose of the first procedure was to fit an ARMA model to the measured time series containing two excited eigen modes. The second procedure contained a bandpass filtering of the time series which meant that in principle only one of the eigen modes were present when the ARMA models was estimated. The second procedure resulted in ARMA models corresponding to each eigen mode.

The first procedure was performed applying the NAG library [3]. The time series were the measured displacements of the two masses of the monopile. For mass no.1 at the top and mass no.2 at the middle the sampling rates were 40 Hz and 60 Hz, respectively. The cut-off frequency was about 12 Hz and the sampling time was 120 seconds.

The satisfactory model order was determined from the reduction in the model error, σ_a^2. When the model error had obtained a minimum, the model order was considered as being satisfactory. Furthermore convergence of the eigen frequencies and the damping ratios were considered as a measure of the applicability of the model. Finally the agreement between the results of the two measured time series also gave some indication of the appropriate model order.

As an example figure 4.5 shows the convergence of the eigen frequencies and damping ratios compared with the development of the model error as a function of the model order. The expected order of the model was an ARMA(4,3) model but both figures show that a considerably higher order is necessary before some kind of convergence has been obtained. In fact it seems as if an ARMA(14,13) model is satisfactory corresponding to a system with 7 degrees of freedom. Any higher order of the models is seen to give no reduction in the model error and the parameters have also somewhat converged. Hence the ARMA(14,13) is chosen as the correct model.

A plausible explanation of this high model order is the shape of the force spectrum. Due to the violated white noise assumption the first eigen mode has not been very strongly represented in the time series. In fact the ratio between the force spectral density at the two eigen frequencies was about a factor 1835. This explains why it has been impossible to identify the first mode in the ARMA models of lower order, see figure 4.5. The mode has simply been hidden in noise.

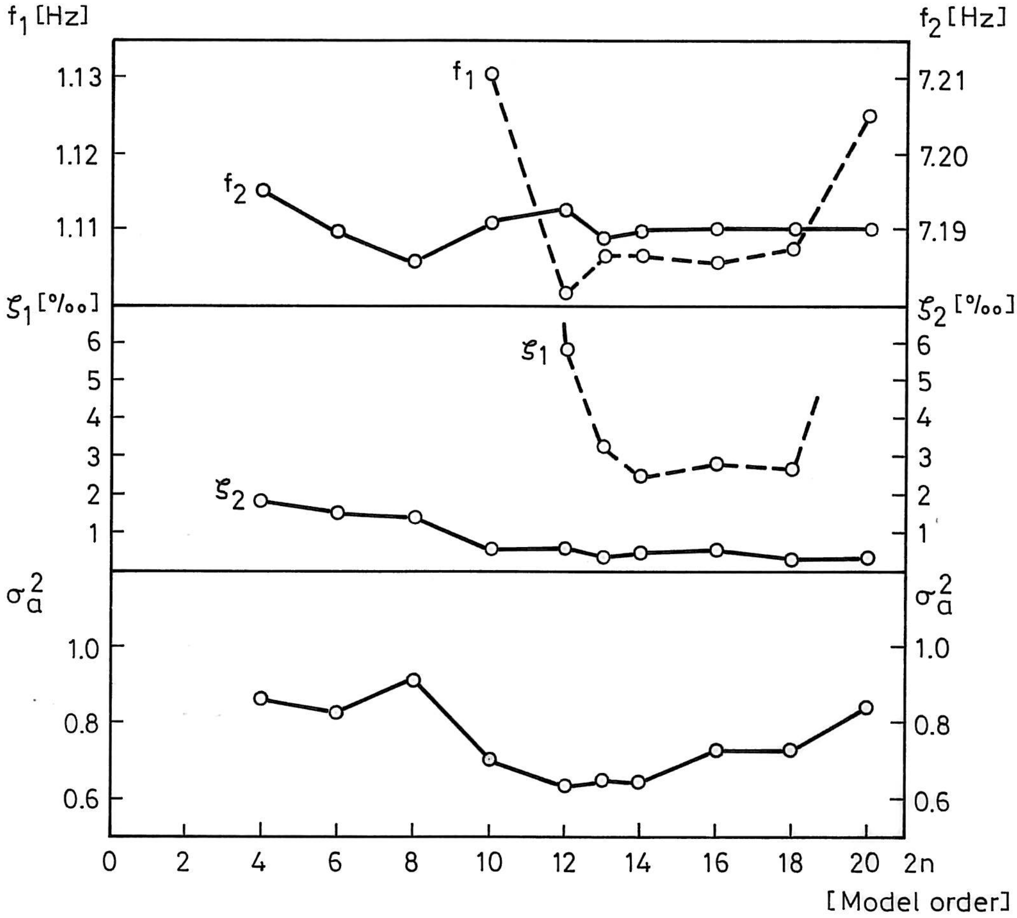

Figure 4.5. The convergence of the eigen frequencies, damping ratios and σ_a^2 as functions of the model order. Factor on σ_a^2 : 10^{-3}. Time series: The displacement response of mass no 2.

The convergence of the eigen frequencies and the damping ratios can be compared in figure 4.5. It is seen that the eigen frequencies converge much more quickly than the damping ratios. This shows the basic fact that the estimation of the damping ratio of a ligthly damped system is more uncertain than the estimation of the eigen frequencies.

The estimated eigen frequencies and damping ratios are shown in table 4.2. From the table it is seen that the estimated eigen frequencies of the two models agree very well. On the other hand it is seen that the agreement between the damping ratios based on the response of the two different masses does not seem to be satisfactory. However, since the statistical uncertainty is not known this might be a wrong conclusion. The statistical uncertainty has not been computed because the NAG-routine failed to give information about the covariance matrix of the ARMA parameters. The different sampling rates may also have had influence on the results.

	f_1 Hz	ζ_1	f_2 Hz	ζ_2
ARMA(14,13) for the displacement of mass 1.	1.1054	0.00127	7.1921	0.00065
ARMA(14,13) for the displacement of mass 2.	1.1070	0.00256	7.1900	0.00045

Table 4.2. Computed eigen frequencies and damping ratios from the estimated ARMA parameters.

The second procedure of the ARMA modelling was to filter the time series before an ARMA model was estimated. This procedure ensured that only one mode was present in the time series. The purpose was to obtain a stronger representation of the first eigen mode and to reduce the proper order of the ARMA models.

The filtering was done by a Yulewalker filter of order 10, available in the software package MATLAB [11]. The sampling frequency of the time series was 50 Hz and the cut-off frequency about 12 Hz. The sampling time was 120 seconds. The expected model was an ARMA(2,1) model because only one mode was supposed to be present in the signals. However, it was found that an ARMA(4,3) model might also be appropriate while all the models of higher order did not provide reasonable results or any substantial reduction in the model error σ_a^2, see figure 4.6. Thus these models of higher order were considered as overfitted models.

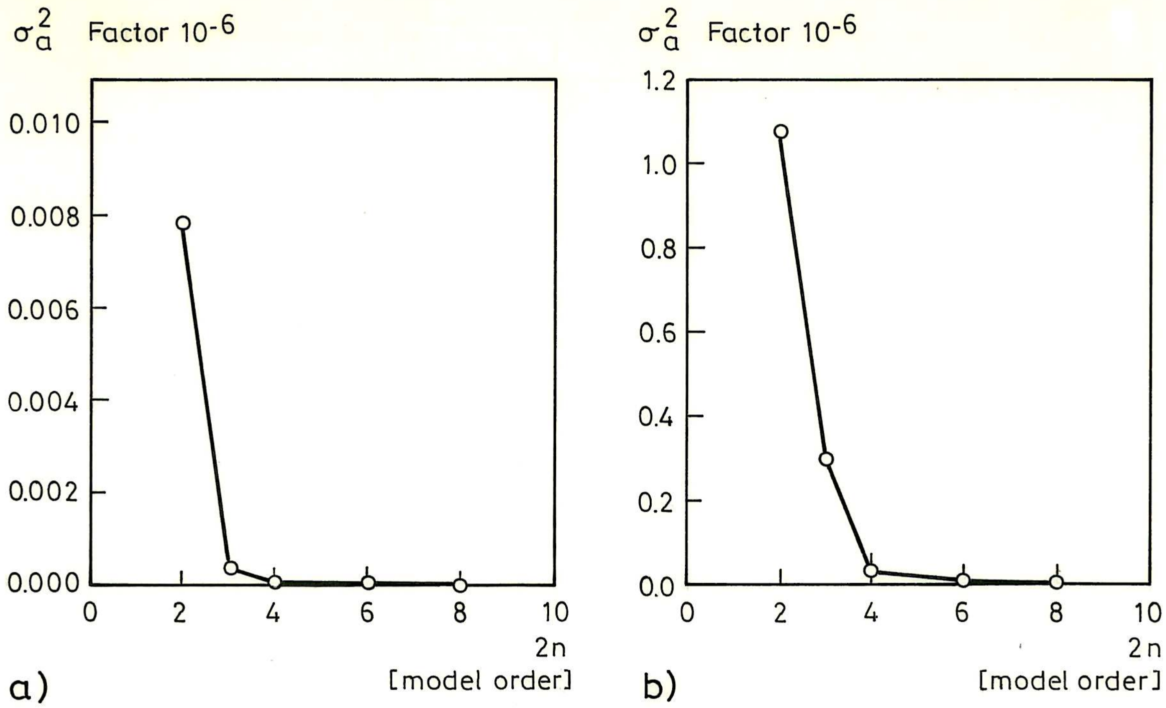

Figure 4.6. The model error expressed by σ_a^2. a) First eigen mode b) Second eigen mode. Time series: Displacement response of mass no 2.

In fact the development of σ_a^2 as a function of the order of the model showed that when the time series was filtered to obtain only the second mode the reduction of σ_a^2 of the ARMA(2,1) and ARMA(4,3) model was about. When the first eigen mode was considered this ratio was about 3000 to 4500 in magnitude. This indicates that the correct model of the time series of the first eigen mode is an ARMA(4,3) model while the second eigen mode may be modelled by an ARMA(2,1) model.

The explanation of the different models is the force spectrum which primarily led to an excitation of the second eigen mode while the first eigen mode was only weakly excited and consequently uncertain to any interpretation. Furthermore, the applied filter was not able to completely eliminate the influence of the second mode in the time series where only the first mode was intended to be present. The mean values and the standard deviations of the eigen frequencies and the damping ratios have been computed by simulation of the sample space of the ARMA parameters, see table 4.3. The ARMA parameters were assumed to be normally distributed with some limitations. The limitations appeared because negative values of the eigen frequencies and the damping ratios were not allowed. The eigen frequencies were assumed to be normally distributed and the damping ratios were assumed to follow a logarithmic normally distribution. This was justified by a χ^2-test. The statistical analysis showed that the eigen frequency and damping ratio of a given mode were uncorrelated.

Time series	ARMA Model	f Hz		ζ	
Mode 1					
Displ.	(2,1)	1.1116	±0.0019	0.00109	±0.00065
Mass 1	(4,3)	1.1069	±0.0001	0.00189	±0.00101
Mode 1					
Displ.	(2,1)	1.1129	±0.0020	0.00294	±0.00154
Mass 2	(4,3)	1.0964	±0.0001	0.00118	±0.00083
Mode 2					
Displ.	(2,1)	7.1935	±0.0008	0.00005	±0.00003
Mass 1	(4,3)	7.1906	±0.0000	0.00066	±0.00000
Mode 2					
Displ.	(2,1)	7.1938	±0.0008	0.00007	±0.00004
Mass 2	(4,3)	7.1813	±0.0301	0.00274	±0.00115

Table 4.3. Computed eigen frequencies and damping ratios from the estimated ARMA parameters with standard deviations.

It is seen that the ARMA models of the two analysed time series can now be compared by the sample values of the mean values and standard deviations of the eigen frequencies and damping ratios given in table 4.3.

If the ARMA(4,3) model is chosen as the proper model of the time series containing the first eigen mode a deviation between the estimated mean values of the parameters are found depending on whether the response was measured at mass 1 or 2. The deviation between the eigen frequencies is about 1% while the deviation between the damping ratios is about 60%. However, the latter is covered by the standard deviation corresponding to a coefficient of variation about 50-70% of the damping ratio. The deviation between the eigen frequencies cannot be directly explained by the uncertainty of the parameter. A possible explanation may be insufficient covergence criteria of the estimations of the ARMA models. The bandpass filtering may also have had some influence.

It is seen that if the ARMA(2,1) model is chosen as the correct model of the time series containing the second eigen mode, the estimated values of the eigen frequency and damping ratio are almost independent of whether the time series has been chosen as the response of mass 1 or 2. This means that the second eigen frequency and damping ratio have been uniquely determined for this model. With respect to the ARMA(4,3) model of the two time series it is seen from table 4.3 that neither the eigen frequencies nor the damping ratios agree. Nevertheless for the time series of the displacement of mass 1 a comparison shows that the results of the ARMA(4,3) model agree very well for the second mode. Consequently, the

ARMA(4,3) model cannot be rejected. The disagreement of the results of the two ARMA(4,3) models might be due to insufficient convergence.

Having discussed the proper models of the two procedures it is now possibly to compare the results and discuss the advantages and disadvantages of the two procedures.

A comparison of the estimates of the eigen frequencies shows that they deviate about 1% with a coefficient of variation of approximately 0.0% of the ARMA(2,1) or ARMA(4,3) models. This is quite acceptable since the deviations are likely to be covered by the uncertainties of the ARMA(14,13) models which the applied ARMA algorithm was not able to compute.

From the second procedure it was found that the damping ratios of the first mode had a variation coefficient of about 50%. The deviations between the damping ratios of the two procedures were about 54% or up to 116% depending on the choice of reference value. Due to the high uncertainty this deviation does not indicate any significant error between the two procedures especially because the deviations are ambiguous. However, with respect to the second mode this is not the case. A comparison of the two procedures shows that the first procedure gave estimates of the second damping ratio which exceeded the estimates of ARMA(2,1) model of the second procedure by a factor 10. On the other hand, if the ARMA(4,3) model was chosen as the proper model of the second procedure the damping estimates deviate less than 2% for the results of the time series of the displacement of mass 1 while the estimates from the time series of the displacement of mass 2 disagree completely. Hence some confusion exists with respect to the proper model order and the determination of the second damping ratio. The explanation might be an insufficient convergence criterion. This should have been studied further if any final conclusion were to be made.

The damping estimate of the first eigen mode has also been compared with the results of a free decay, see Jensen [10]. Here the damping ratio was found to $\zeta_1 = 0.00108 \pm 0.00002$ which lies within the uncertainty of the estimates of the ARMA models provided by the second procedure.

The comparison of the two procedures has shown that both procedures can be applied since the parameter estimates computed by the two procedure agree quite well with the exception of the second damping ratio. However, the second procedure seems to be the most advantageous because of the considerably smaller order of the most appropriate model. The small model order means that the estimation requires considerably less computer time and it means that the ARMA estimation algorithm almost always succeeds in computing the uncertainty of the ARMA parameters. Finally, due to the low model order it is a practicable task to compute the statistics of the eigen frequencies and the damping ratios by simulation of the sample space of the AR parameters.

5 CONCLUSION

From the theoretically derived expressions it has been possible to compute reliable statistical estimates of the eigen frequencies and the damping ratio of a lightly and heavily damped vibrating system.

The statistical estimates were found by simulation of the sample space of the AR parameters. This can be done with moderate computer costs for ARMA models of low order. The statistical analysis showed that the estimates of the eigen frequency and the damping ratio of a given mode were uncorrelated. The eigen frequencies of the lightly damped system were estimated with variation coefficients about 0% whereas the variation coefficient was found to be about 17% for the heavily damped system. In both cases the damping ratio showed a variation coefficient about 50%. Here it can be noticed that estimates of the damping ratio can be found with a variation coefficient as small as 1% from a free decay, see Jensen [10]. It should be further studied whether this significant difference in uncertainty is due the to random nature of the vibration or is caused by the modelling of the damping mechanisms in the structure.

The appropiate model orders of the ARMA models were found from from the development in the model error of the ARMA model as a function of the model order. Initial estimates of the model order was found from theoretical models and FFT analysis. The convergence of the eigen frequencies and the damping ratios were also applied as an indicator of the appropriate model order. The comparison of the estimates of different time series of the same structure also gave valuable information. The development of a objective choice of the model order may be subject to further research.

More knowledge about the convergence demands in the estimation of the ARMA parameters should also be studied since it is of significant importance for the final estimates of the eigen frequencies and the damping ratios.

The application of ARMA models has shown that the most advantageous procedure to estimate the eigen frequencies and the damping ratios is to ensure that only one mode is present in the time series. This have been done with succes when one of two modes was very weakly excited. However, if two eigen modes are coupled close together this may not be possible. Thus the application of ARMA models in these cases may be subject to further study.

ACKNOWLEDGEMENTS

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Appendix A

The equivalent covariance properties can be derived directly from the state space formulation in the continous time domain (3.2) and the discrete time domain (3.6).

In the continuous time domain it follows from (3.2) that:

$$\frac{d\overline{x}(t)}{dt}\,\overline{x}^T(s) = \overline{\overline{A}}\,\overline{x}(t)\,\overline{x}^T(s) + \overline{b}\,a(t)\,\overline{x}^T(s) \tag{A1}$$

$$E[a(t)\overline{x}(s)] = \overline{0}, \quad s < t \tag{A2}$$

From the relation between the cross correlation matrix of $x(t)$ and $\dot{x}(t)$ and the auto covariance matrix of $x(t)$:

$$\overline{\overline{C}}_{x\dot{x}}(\tau) = \frac{d\overline{\overline{C}}_{xx}(\tau)}{dt} \tag{A3}$$

it follows that:

$$\frac{d}{dt}\overline{\overline{C}}_{xx}(t-s) = \overline{\overline{A}}\,\overline{\overline{C}}_{xx}(t-s) \tag{A4}$$

which has the solution:

$$\begin{aligned}\overline{\overline{C}}_{xx}(\tau) &= \exp(\overline{\overline{A}}\tau)\overline{\overline{C}}_{xx}(0)\\ &= \exp(\overline{\overline{A}}s\Delta)\overline{\overline{C}}_{xx}(0), \quad \tau = s\Delta\end{aligned} \tag{A5}$$

In the same way it is possible to obtain an expression for the covariance function in the discrete time domain:

$$\overline{x}_{k+1}\overline{x}^T_{k-s} = \exp(\overline{\overline{A}}\Delta)\overline{x}_k\overline{x}^T_{k-s} + \overline{u}_k\overline{x}^T_{k-s} \tag{A6}$$

$$E[\overline{u}_k\overline{x}_{k-s}] = 0, \quad k > k - s \tag{A7}$$

$$\begin{aligned}\overline{\overline{C}}_{xx}(s+1) &= \exp(\overline{\overline{A}}\Delta)E[\overline{x}_k\overline{x}^T_{k-s}]\\ &= \exp(\overline{\overline{A}}\Delta)\overline{\overline{C}}_{xx}(s)\end{aligned} \tag{A8}$$

This leads to the solution:

$$\overline{\overline{C}}_{xx}(s) = \exp(\overline{\overline{A}}s\Delta)\overline{\overline{C}}_{xx}(0), \quad s = 1, 2 \ldots \tag{A9}$$

Now, if (A9) is compared with (A5) it is seen that the two covariance functions are equivalent. This means that the ARMA(*2n,2n-1*) model should be chosen when an n degrees of freedom system is considered.

Appendix B

Lemma: Assume that $\overline{\overline{A}}$ has the eigen values given by the diagonal matrix (λ_i) then $\exp(\overline{\overline{A}}\Delta)$ will have the eigen values given by the diagonal matrix $(\exp(\lambda_i\Delta))$.

Proof: If $\overline{\overline{A}}$ has the eigen values (λ_i) then the matrix $(\overline{\overline{A}})^n$ will have the eigen values $(\lambda_i{}^n)$. This means that if:

$$(\exp(\overline{\overline{A}}\Delta) - \overline{\overline{I}}\exp(\lambda\Delta))\overline{x} = 0 \tag{B1}$$

is expanded into a Taylor expansion:

$$(\overline{\overline{I}} + \overline{\overline{A}}\Delta + \frac{1}{2!}(\overline{\overline{A}}\Delta)^2 + \cdots \frac{1}{n!}(\overline{\overline{A}}\Delta)^n - \overline{\overline{I}} - \overline{\overline{I}}\Delta\lambda - \frac{1}{2!}(\overline{\overline{I}}\Delta\lambda)^2 - \cdots - \frac{1}{n!}(\overline{\overline{I}}\Delta\lambda)^n)\overline{x} = 0 \tag{B2}$$

it is seen that the Lemma is satisfied.

Appendix C

The expression of the auto spectrum of x_t can be derived from the ARMA model of the time series:

$$x_t - \Phi_1 x_{t-} \cdots - \Phi_{2n} x_{(t-2n)} = a_t - \Theta_1 a_{(t-1)} \cdots - \Theta_{(2n-1)} a_{(t-2n+1)} \tag{C1}$$

If the z-transformed defined by:

$$Y(z) = \sum_{t=-\infty}^{+\infty} y_t z^{-t} \tag{C2}$$

if used in (C1) the following transfer function is obtained:

$$H(z) = \frac{1 - \Theta_1 z^{-1} \cdots - \Theta_{2n-1} z^{-(2n-1)}}{1 - \Phi_1 z^{-}1 \cdots - \Phi_{2n} z^{-2n}} \tag{C3}$$

Now if z is replaced by $\exp(i\omega)$ one obtains the transfer function given in the frequency domain:

$$H(\omega) = \frac{\exp(i\omega)(1 - \Theta_1 \exp((2n-1)i\omega) \cdots - \Theta_{2n-1})}{\exp(2ni\omega) - \Phi_1 \exp((2n-1)i\omega) - \Phi_{2n}} \tag{C4}$$

If (C4) is multiplied with the conjugated of (C4) the squared magnitude of $H(\omega)$ is determined to be:

$$\|H(\omega)\|^2 = \frac{\|\exp((2n-1)i\omega) - \Theta_1 \exp((2n-2)i\omega) - \cdots - \Theta_{2n-1}\|^2}{\|\exp(2ni\omega) - \Phi_1 \exp((2n-1)i\omega) - \cdots - \Phi_{2n}\|^2} \tag{C5}$$

Now the response spectrum S_{xx} can be determined by

$$S_{xx} = \|H(\omega)\|^2 S_{aa} \tag{C6}$$

with $S_{aa} = {\sigma_a}^2/(2\pi)$. However, the above relations only apply to the double-sided discrete frequency domain and the aim is a one-sided continuously auto-spectrum. This is obtained due to:

$$Y(\exp(i\omega\Delta)) = \frac{1}{\Delta}\tilde{Y}(\omega) \tag{C7}$$

where Y and $\tilde{Y}$ are the discrete and the continuous FFT-transformed respectively. This leads to the expression of the continuous one-sided auto-spectrum of a_t:

$$\tilde{S}_{aa}(f) = 2(2\pi)\Delta S_{aa} \qquad 0 \leq f \leq \frac{1}{\Delta} \tag{C8}$$

If $(C8)$ is applied in $(C6)$ with $(C5)$ the final expression for the one-sided continuous auto-spectrum is found as:

$$S_{xx}(f) = 2\Delta{\sigma_a}^2 \frac{\|\exp((2n-1)i\Delta f 2\pi) - \Theta_1 \exp((2n-2)i\Delta f 2\pi) - \cdots \Theta_{2n-1}\|^2}{\|\exp(2ni\Delta f 2\pi) - \Phi_1 \exp((2n-1)i\Delta f 2\pi) - \cdots - \Phi_{2n}\|^2}$$

$$0 \leq f \leq \frac{1}{2\Delta} \tag{C9}$$

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